

RATIONAL FACTORIZATIONS OF COMPLETELY POSITIVE MATRICES

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ABSTRACT. In this note it is proved that every rational matrix which lies in the interior of the cone of completely positive matrices also has a rational cp-factorization.

1. INTRODUCTION

The cone of completely positive matrices is central to copositive programming, see [3] and also to several topics in matrix theory, see [1]. However, so far, this cone is quite mysterious, many basic questions about it are open. In [2] Berman, Dür, and Shaked-Monderer ask: *Given a matrix $A \in \mathcal{CP}_n$ all of whose entries are integral, does A always have a rational cp-factorization?*

The *cone of completely positive matrices* is defined as the convex cone spanned by symmetric rank-1-matrices xx^\top where x lies in the nonnegative orthant $\mathbb{R}_{\geq 0}^n$:

$$\mathcal{CP}_n = \text{cone}\{xx^\top : x \in \mathbb{R}_{\geq 0}^n\}.$$

A *cp-factorization* of a matrix A is a factorization of the form

$$A = \sum_{i=1}^m \alpha_i x_i x_i^\top \quad \text{with } \alpha_i \geq 0 \text{ and } x_i \in \mathbb{R}_{\geq 0}^n, \quad \text{for } i = 1, \dots, m.$$

We talk about a *rational cp-factorization* when the α_i 's are rational numbers and when the x_i 's are rational vectors. Of course, in a rational cp-factorization we can assume that the x_i 's are integral vectors.

In this note we prove the following theorem:

Theorem 1.1. *Every rational matrix which lies in the interior of the cone of completely positive matrices has a rational cp-factorization.*

So to fully answer the question of Berman, Dür, and Shaked-Monderer, it remains to consider the boundary of \mathcal{CP}_n .

2. PROOF OF THEOREM 1.1

For the proof we will need a classical result from simultaneous Diophantine approximation, a theorem of Dirichlet, which we state here. One can find a proof of Dirichlet's theorem for example in the book [4, Theorem 5.2.1] of Grötschel, Lovász, and Schrijver.

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Theorem 2.1. *Let $\alpha_1, \dots, \alpha_n$ be real numbers and let ε be a real number with $0 < \varepsilon < 1$. Then there exist integers p_1, \dots, p_n and a natural number q with $1 \leq q \leq \varepsilon^{-n}$ such that*

$$\left| \alpha_i - \frac{p_i}{q} \right| \leq \frac{\varepsilon}{q} \quad \text{for all } i = 1, \dots, n.$$

The next lemma collects standard, easy-to-prove facts about convex cones. Let E be a Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Let $K \subseteq E$ be a *proper convex cone*, which means that K is closed, has a nonempty interior, and satisfies $K \cap (-K) = \{0\}$. Its *dual cone* is defined as $K^* = \{y \in E : \langle x, y \rangle \geq 0 \text{ for all } x \in K\}$.

Lemma 2.2. *Let $K \subseteq E$ be a proper convex cone. Then,*

$$(1) \quad \text{int}(K) = \{x \in E : \langle x, y \rangle > 0 \text{ for all } y \in K^* \setminus \{0\}\},$$

where $\text{int}(K)$ is the topological interior of K , and

$$(2) \quad K^* = (\text{cl}(K))^*,$$

where $\text{cl}(K)$ is the topological closure of K .

We need some more notation: With \mathcal{S}^n we denote the vector space of symmetric matrices with n rows and n columns which is a Euclidean space with inner product $\langle A, B \rangle = \text{Trace}(AB) = \sum_{i,j=1}^n A_{ij}B_{ij}$. The *cone of copositive matrices* is the dual cone of \mathcal{CP}_n :

$$\mathcal{COP}_n = \mathcal{CP}_n^* = \{B \in \mathcal{S}^n : \langle A, B \rangle \geq 0 \text{ for all } A \in \mathcal{CP}_n\}.$$

Its interior equals

$$\text{int}(\mathcal{COP}_n) = \{B \in \mathcal{S}^n : \langle B, xx^\top \rangle > 0 \text{ for all } x \in \mathbb{R}_{\geq 0}^n \setminus \{0\}\}.$$

We also define the following rational subcone of \mathcal{CP}_n :

$$\tilde{\mathcal{CP}}_n = \text{cone}\{vv^\top : v \in \mathbb{Z}_{\geq 0}^n\}.$$

We prepare the proof of the paper's main result by two lemmata which might be useful facts themselves.

Lemma 2.3. *The set*

$$\mathcal{R} = \{B \in \mathcal{S}^n : \langle B, vv^\top \rangle \geq 1 \text{ for all } v \in \mathbb{Z}_{\geq 0}^n \setminus \{0\}\},$$

is contained in the interior of the cone of copositive matrices \mathcal{COP}_n .

Proof. Since the set of nonnegative rational vectors $\mathbb{Q}_{\geq 0}^n$ lies dense in the nonnegative orthant $\mathbb{R}_{\geq 0}^n$, we have the inclusion $\mathcal{R} \subseteq \mathcal{COP}_n$. Suppose for contradiction that the set on the left is not contained in $\text{int}(\mathcal{COP}_n)$: There is a matrix B with $\langle B, vv^\top \rangle \geq 1$ for all $v \in \mathbb{Z}_{\geq 0}^n \setminus \{0\}$ and there is a nonzero vector $x \in \mathbb{R}_{\geq 0}^n$ with $\langle B, xx^\top \rangle = 0$.

By induction on n (and reordering if necessary) we may assume that all entries of x are strictly positive, $x_i > 0$ for all $i = 1, \dots, n$, since otherwise, we can reduce the situation to the case of smaller dimension by considering a suitable submatrix of B .

Hence, the vector x lies in the interior of the nonnegative orthant. Therefore, and because $B \in \mathcal{COP}_n$, we have for every vector $y \in \mathbb{R}^n$ and $\varepsilon > 0$ sufficiently small the inequality

$$0 \leq \frac{1}{\varepsilon}(x + \varepsilon y)^\top B(x + \varepsilon y) = 2x^\top B y + \varepsilon y^\top B y$$

and similarly

$$0 \leq \frac{1}{\varepsilon}(x - \varepsilon y)^\top B(x - \varepsilon y) = -2x^\top B y + \varepsilon y^\top B y$$

From this, equality $x^\top B = 0$ follows. From this, we also see that B is positive semidefinite. This implies that

$$(\alpha x + y)^\top B(\alpha x + y) = y^\top B y \quad \text{for } \alpha \in \mathbb{R} \text{ and } y \in \mathbb{R}^n.$$

We apply Dirichlet's approximation theorem, Theorem 2.1 to the vector x and to $\varepsilon \in (0, 1)$. We obtain a vector $p = (p_1, \dots, p_n)$ and a natural number q . Since $x_i > 0$ we may without loss of generality assume that $p_i \geq 0$. Thus, by the assumption $B \in \mathcal{R}$, we have $\langle B, pp^\top \rangle \geq 1$.

Define

$$y = qx - p \quad \text{where} \quad \|y\|_\infty \leq \varepsilon.$$

Since B is positive semidefinite, there is a constant C such that $y^\top B y \leq C\|y\|_\infty^2$ for all $y \in \mathbb{R}^n$. Putting everything together we get

$$1 \leq \langle B, pp^\top \rangle = (qx - y)^\top B(qx - y) = y^\top B y \leq C\|y\|_\infty^2 \leq C\varepsilon^2,$$

which yields a contradiction for small enough values of ε . \square

Lemma 2.4. *Let A be a completely positive matrix which lies in the interior of \mathcal{CP}_n and let λ be a sufficiently large positive real number. Then the set*

$$\mathcal{P}(A, \lambda) = \{B \in \mathcal{S}^n : \langle A, B \rangle \leq \lambda, \langle B, vv^\top \rangle \geq 1 \text{ for all } v \in \mathbb{Z}_{\geq 0}^n \setminus \{0\}\}$$

is a full-dimensional polytope.

Proof. For sufficiently large λ a sufficiently small ball around a suitable multiple of A is contained in $\mathcal{P}(A, \lambda)$, which shows that $\mathcal{P}(A, \lambda)$ has full dimension.

By the theorem of Minkowski and Weyl, see for example [5, Corollary 7.1c], polytopes are exactly bounded polyhedra. So it suffices to show that the set $\mathcal{P}(A, \lambda)$ is a bounded polyhedron.

First we show that $\mathcal{P}(A, \lambda)$ is bounded: For suppose not. Then there is $B_0 \in \mathcal{P}(A, \lambda)$ and $B_1 \in \mathcal{S}^n$, with $B_1 \neq 0$, so that the ray $B_0 + \alpha B_1$, with $\alpha \geq 0$, lies completely in $\mathcal{P}(A, \lambda)$. In particular $\langle B_1, vv^\top \rangle \geq 0$ for all $v \in \mathbb{Z}_{\geq 0}^n$. Hence, B_1 lies in the dual cone of $\tilde{\mathcal{CP}}_n$. On the other hand $\langle A, B_1 \rangle \leq 0$. Hence, by Lemma 2.2 (1), $B_1 \notin \mathcal{COP}_n \setminus \{0\}$, but by Lemma 2.2 (2),

$$\tilde{\mathcal{CP}}_n^* = (\text{cl}(\tilde{\mathcal{CP}}_n))^* = \mathcal{CP}_n^* = \mathcal{COP}_n,$$

so $B_1 = 0$, yielding a contradiction.

Now we show that $\mathcal{P}(A, \lambda)$ is a polyhedron: For suppose not. Then there is a sequence $v_i \in \mathbb{Z}_{\geq 0}^n \setminus \{0\}$ of infinitely many pairwise different nonzero lattice vectors so that there are $B_i \in \mathcal{P}(A, \lambda)$ with $\langle B_i, v_i v_i^\top \rangle = 1$. Since $\mathcal{P}(A, \lambda)$ is compact, there exists a subsequence B_{i_j} which converges to $B^* \in \mathcal{P}(A, \lambda)$. Define the sequence $u_{i_j} = v_{i_j} / \|v_{i_j}\|$ which lies in the compact set $\mathbb{R}_{\geq 0}^n \cap S^{n-1}$ where S^{n-1} denotes the unit sphere. Hence there is a subsequence converging to $u^* \in S^{n-1}$, in particular $u^* \neq 0$. Denote the indices of this subsequence with k , then

$$1 = \langle B_k, v_k v_k^\top \rangle = \|v_k\|^2 \langle B_k, u_k u_k^\top \rangle.$$

When k tends to infinity, the squared norms $\|v_k\|^2$ tend to infinity as well, since we use infinitely many pairwise different lattice vectors and there exist only finitely

many lattice vectors up to some given norm. So $\langle B_k, u_k u_k^\top \rangle$ tends to $\langle B^*, u^*(u^*)^\top \rangle = 0$, and by Lemma 2.3 we obtain a contradiction. \square

Now we prove the main result and finish the paper.

Proof of Theorem 1.1. Let A be matrix having rational entries only and lying in the interior of the cone of completely positive matrices. Then $\mathcal{P}(A, \lambda)$ is a polytope according to the previous lemma. We minimize the linear functional $B \mapsto \langle A, B \rangle$ over $\mathcal{P}(A, \lambda)$. The minimum is attained at one of the polytopes' vertices, $B^* \in \mathcal{P}(A, \lambda)$. Then we choose those lattice vectors $v_i \in \mathbb{Z}_{\geq 0}^n$, with $i = 1, \dots, m$ for which equality $\langle B^*, v_i v_i^\top \rangle = 1$ holds. Because of the minimality of $\langle A, B^* \rangle$ it follows

$$(3) \quad A \in \text{cone}\{v_i v_i^\top : i = 1, \dots, m\}.$$

Otherwise, see for example [5, Theorem 7.1], we find a separating linear hyperplane orthogonal to C separating A and $\text{cone}\{v_i v_i^\top : i = 1, \dots, m\}$:

$$\langle C, A \rangle < 0 \quad \text{and} \quad \langle C, v_i v_i^\top \rangle \geq 0 \quad \text{for all } i = 1, \dots, m.$$

Then for sufficiently small $\mu > 0$ we would have

$$B^* + \mu C \in \mathcal{P}(A, \lambda) \quad \text{but} \quad \langle B^* + \mu C, A \rangle < \langle B^*, A \rangle,$$

which contradicts the minimality of $\langle A, B^* \rangle$.

We apply Carathéodory's theorem (see for example [5, Corollary 7.1i]) to (3) and choose a subset $I \subseteq \{1, \dots, m\}$ so that $v_i v_i^\top$ are linearly independent and so that A lies in $\text{cone}\{v_i v_i^\top : i \in I\}$. Since A is a rational matrix and since the $v_i v_i^\top$'s are linearly independent rational matrices, there is a unique choice of rational numbers $\alpha_i \in \mathbb{Q}_{\geq 0}$, with $i \in I$, so that $A = \sum_{i \in I} \alpha_i v_i v_i^\top$ holds, which gives a desired rational cp-factorization. \square

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